APPLICATION OF A CONDITIONAL EXPECTATION RESPONSE SURFACE APPROACH TO PROBABILISTIC FATIGUE

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Abstract
As probabilistically-based design computer programs become more prevalent, the capability to consider random variables not included in the design code will become more desirable and important. This research describes a probabilistic methodology based upon conditional expectation such that additional random variables can be considered for a probabilistic design code without modifying the software. Sensitivity measures have been developed for both the original design (internal) and (external) additional random variables. The value is that developers can quickly and easily assess the significance of new random variables without resorting to code modifications, and users who do not have access to the source code can consider additional random variables pertinent to their problems. The methodology is demonstrated using a probabilistic fatigue application.

Introduction
Specialized probabilistically-based design and analysis computer codes have been developed that are tailored to particular applications such as aerospace, nuclear, or offshore structures. For example, in aerospace, two prominent codes are PROF (Probability of Fracture, Berens et al., 1991) and DARWIN® (Design Assessment of Reliability With INspection, Leverant, 2000, Leverant et al., 2003). These design codes contain specific mechanics models, random variables, and probabilistic methods and are highly optimized for a particular application.

It is inevitable that over time, circumstances may arise in which additional random variables may be needed. For example, a user may identify a new scenario with a scope broader than the intent of the original design code. Also, the developer who is considering expanding the product and must discern which additional random variables to consider can use this technology to quickly make decisions.

Conditional Expectation Methodology
Conditional expectation is a variance reduction sampling method that can significantly reduce the required number of samples for a given accuracy. The random variables are partitioned into two sets: conditional ($\hat{X}$) and control ($\tilde{X}$). The total probability of failure can be determined as the expected value of the probability of failure estimates considering realizations of the conditional variables, with the control variable random. Significant variance reduction is obtained if the variable with the largest variance is designated the control variable. This procedure in effect removes the variance of the control variable from the analysis.
The conditional expectation of the probability of failure is obtained using the formula (Ayyub and Chia, 1992)

$$P_f = E_{\tilde{X}} \left( F_{\hat{X}} \left[ g(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_M, \hat{X}) \leq 0 \right] \right)$$

(1)

where $\tilde{X}_i$ denotes a conditional variable and $\hat{X}$ is the control variable, and $F_{\hat{X}}$ is the cdf of the control variable. Eqn. 1 can be approximated as

$$P_f \approx \frac{1}{N} \sum_{i=1}^{N} P_{f_i}$$

(2)

where $N$ is the number of samples and $P_{f_i} = \left( F_{\hat{X}} \left[ g(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_M, \hat{X}) \leq 0 \right] \right)$ for a given realization of $\hat{X}$.

The variance (var) and the coefficient of variation (COV) of the estimated probability of failure (Ayyub and Chia, 1992; Karamchandani and Cornell, 1991), respectively, are given by

$$\text{var}(\bar{P}_f) = \frac{\sum_{i=1}^{N} (P_{f_i} - \bar{P}_f)^2}{N(N-1)}$$

$$COV(\bar{P}_f) = \frac{\sqrt{\text{var}(\bar{P}_f)}}{\bar{P}_f}$$

(3)

**Generalized Conditional Expectation (GCE)**

Generalized conditional expectation (GCE) is a variance reduction technique with multiple control variables. GCE is derived from Eqn. (1) as

$$P_f = E_{\tilde{X}} \left[ P_f(\tilde{X}, \hat{X}) \right]$$

(4)

where $\hat{X}$ is now a vector of control random variables.

The probability of failure for the given set of control variables in the domain of conditional expectation (Ayyub and Chia, 1992) is given by

$$P_f = E_{\tilde{X}} \left\{ \text{prob}[g(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_M, \hat{X}_1, \hat{X}_2, \ldots \hat{X}_K) < 0] \right\}$$

(5)

The probability expression in Eqn. 6 $\text{prob}[g(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_M, \hat{X}_1, \hat{X}_2, \ldots \hat{X}_K) < 0]$ can be evaluated using any probabilistic method (e.g., numerical simulation, second moment-based methods, etc.).
Sensitivities of conditional and control random variables

Sensitivity measures of the probability of failure with respect to conditional and control variables, e.g., $\frac{\partial P_f}{\partial \mu}$, $\frac{\partial P_f}{\partial \sigma}$ have been developed that can be used to assess the relative importance of the variables. Sensitivities for control variables, however, are dependent on the sensitivities being intrinsic to the probabilistic design code.

Numerical Example

A numerical example is presented using the probabilistic fatigue design code DARWIN to demonstrate the methodology. The following control random variables are considered in DARWIN: initial crack size, life scatter, stress scatter, probability of detection, and time of inspection. In this example, two conditional random variables are considered: disk rotational speed and applied load.

Traditionally, the control variables are designated as those variables with the largest COV. However, in our application the control variables must be those within the probabilistic design code and the conditional variables those new random variables to be considered. Hence, the control variables are those already considered in DARWIN (initial crack size, life scatter, etc.) and the conditional variables are the loading variables external to DARWIN.

The focused set of random variables in DARWIN, initial crack size, life scatter, probability of detection, etc., is expanded to include variables affecting the gas turbine disk stress distribution - disk rotational speed and external load. In effect, the total random variables considered are partitioned into two groups – those contained within DARWIN (control variables), addressed by DARWIN’s probabilistic algorithms, and those outside DARWIN (conditional variables). The probabilistic analysis considering all random variables is computed by conditional expectation with the DARWIN probability of failure approximated by a response surface. This analysis involves coupled finite element analysis and probabilistic fatigue. The methodology is applied to a simple analysis for which a comparison solution is known and a more realistic engineering analysis.

Figure 1 shows the finite element model of the cross section of a square titanium ring under rotation with an external load. The rotation alternates from zero to 6800 RPM. This problem is outlined in the FAA advisory circular AC-33.14-1 (Federal Aviation Administration, 2001) The geometry consists of a ring disk under cyclic loading of 20,000 cycles. The disk is subjected to centrifugal loading (maximum speed of 6800 RPM) and an external load of 50 Mpa (7.25 ksi) applied on the outer diameter to simulate blade loading. A surface crack with a 1-1 aspect ratio is located on the inner bore. Failure is defined as the crack reaching a critical size, or equivalently, the stress intensity exceeding the fracture toughness.

The statistical characteristics for the problem solved are given in Table 1. The initial size of the surface crack is defined by an industry-determined exceedance curve. (Aerospace
Industries Association Rotor Integrity Sub-Committee, 1997). This random variable is internal to DARWIN. The RPM and external pressure are conditional variables external to DARWIN.

The schematic for implementing this methodology is shown in Figure 2. The control software implements the GCE. This involves multiple calls to DARWIN with different realizations of the conditional variables. Because the conditional variables affect the finite element (FE) model, the FE analysis and a stress translator ANS2NEU are called for each realization.

Any technique to compute the expected value of DARWIN results, e.g., solve eqn. 5, can be used. However, for computational efficiency considerations, a response surface of the $P_f$ from DARWIN versus the conditional random variables is developed.

The results for the expected value of probability of failure and the variance are shown in table 2. The $P_f$ is determined three ways: 1) using standard Monte Carlo (no GCE) with an analytical Matlab solution that mimics DARWIN computation (serves as a benchmark), 2) using CGE as outlined in Figure 2 without response surface, and 3) using CGE as outlined in Figure 2 with response surface. There is good agreement among all methods. Table 3 indicates the sensitivities with respect to the mean and standard deviation of each variable. Again, there is good agreement among all methods.

![Figure 1. Geometry of the Advisory Circular AC-33.14-1](image)

Element type - Plane42

1444 elements

Speed $\omega$

6800 rpm
Figure 2. Schematic of procedure for considering external random variables

<table>
<thead>
<tr>
<th>Type</th>
<th>Name</th>
<th>Distribution</th>
<th>Parameter 1 (Mean)</th>
<th>Parameter 1 (COV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional</td>
<td>Omega</td>
<td>Normal</td>
<td>712.35</td>
<td>0.05</td>
</tr>
<tr>
<td>Conditional</td>
<td>Pressure</td>
<td>Normal</td>
<td>7250</td>
<td>0.1</td>
</tr>
<tr>
<td>Control</td>
<td>Initial crack size</td>
<td>Exceedance curve</td>
<td>---</td>
<td>---</td>
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</table>

Table 1. Statistical characteristics of random variables

<table>
<thead>
<tr>
<th>Method</th>
<th>No. of Samples (N)</th>
<th>Mean POF</th>
<th>Variance Var (P_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo (analytical)</td>
<td>1000 Matlab</td>
<td>0.21300</td>
<td>6.702e-4</td>
</tr>
<tr>
<td>GCE (Ansys (MC) and Darwin)</td>
<td>1000 FE &amp; DARWIN</td>
<td>0.20667</td>
<td>1.869e-5</td>
</tr>
<tr>
<td>GCE(Ansys (RS) and Darwin)</td>
<td>9 FE &amp; DARWIN with 100,000 MC RS Samples</td>
<td>0.20760</td>
<td>6.5e-3</td>
</tr>
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</table>

Table 2. Expected value of probability of failure and variance
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Random variable</th>
<th>Monte Carlo (Analytical Solution)</th>
<th>GCE Ansys DARWIN (Analytical Formula)</th>
<th>GCE Ansys DARWIN (Finite Difference)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Omega</td>
<td>0.316E-2</td>
<td>0.334E-2</td>
<td>0.351E-2</td>
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<tr>
<td></td>
<td>Pressure</td>
<td>0.8E-4</td>
<td>0.74E-4</td>
<td>0.87E-4</td>
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<tr>
<td>Standard deviation</td>
<td>Omega</td>
<td>0.123E-2</td>
<td>0.143E-2</td>
<td>0.124E-2</td>
</tr>
<tr>
<td></td>
<td>Pressure</td>
<td>0.5E-4</td>
<td>0.609E-4</td>
<td>0.552E-4</td>
</tr>
</tbody>
</table>

Table 3. Sensitivity of mean and standard deviation of conditional random variables.

Summary and Conclusions

A methodology is presented and demonstrated that can be used to consider the affect of additional random variables on a probabilistic design code without modifying the design code. The methodology uses the generalized conditional expectation approach. This technology may be useful to the user who wishes to apply the code in a new scenario, which requires additional random variables. Also, the developer who is considering expanding the applicability of the product can assess the relative importance of additional random variables.

Acknowledgements

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References


