Fracture mechanics-based probabilistic life prediction of components with large numbers of inherent material anomalies

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ABSTRACT: Material anomalies are occasionally introduced in the commercial grade alloys used in aircraft gas turbine engine rotating components. If undetected during manufacturing or subsequent field inspection, the anomalies can lead to uncontained engine failures. Over the past several years, a probabilistic framework has been developed to predict the risk of fracture associated with rotors and disks in commercial aircraft engines. The framework was originally developed for titanium materials, where inherent anomalies may be present in the form of brittle alpha phase particles that are assumed to form growing cracks during the first cycle of applied load. Since the anomaly occurrence rate associated with titanium is extremely small, component failure probability can be approximated as the sum of the failure probabilities of subregions (zones) of approximately equal risk. In contrast, some gas turbine materials may exhibit a larger number of anomalies that are either introduced during component manufacturing or are inherently associated with material processing. When these materials are used, a single disk could contain a number of anomalies with unequal crack formation periods, so the existing probabilistic framework is no longer valid. In this paper, a probabilistic fracture mechanics methodology is presented for risk assessment of components with relatively large numbers of material anomalies. It centers on the zone-based probabilistic framework originally developed for titanium materials with hard alpha anomalies, and is extended for application to other alloys. The methodology is presented and illustrated for an aircraft gas turbine engine disk. The results can be applied to fracture-mechanics-based probabilistic life prediction of alloys with large numbers of material anomalies.

1 INTRODUCTION

Gas turbine engines used in the commercial aircraft industry rely on rotors and disks to support the networks of airfoil shaped blades used to force air and fuel to produce thrust. These engines operate at high speeds that may provide a failed disk with sufficient energy to penetrate the external engine cowling. The consequences of this event can be catastrophic, including loss of life and loss of the aircraft. For out of condition events, this issue is currently addressed by the safe-life methodology used for the design and life management aircraft turbine rotors.

Some uncontained failures can be attributed to the occurrence of rare material anomalies that may form during the manufacturing process (NTSB 1990). For example, the premium grade titanium materials used for fan and compressor disks may contain brittle (hard alpha) anomalies that form during the triple vacuum arc melting process (Leverant *et al.* 2000). These anomalies may occur anywhere within a billet and may even change shape during forging.

Fortunately, these inherent anomalies are extremely rare in most gas turbine materials. For example, the titanium alloys commonly used in commercial gas turbine engine rotors and disks have an occurrence rate of less than one anomaly per million pounds of material for a median anomaly size of 20 mils (Aerospace Industries Association 1997). Due to the rare occurrence and high consequence of failure associated with these anomalies, a probabilistic approach is well suited to the life management of rare material anomalies. Guidelines for this enhancement to the safe-life approach are included in a recently released aircraft advisory circular devoted to rotors and disks with inherent anomalies (Federal Aviation Administration 2001). This enhancement is intended to supplement, not replace, the current safe-life methodology.

Uncontained engine failures have also been attributed to induced anomalies (NTSB 1996). The anomalies are introduced during manufacturing and handling operations, and are typically found on machined surfaces. A number of researchers have reported disk failures that initiated at the interior surfaces of bolt holes (e.g., Melis and Zaretsky 1999, Shlyannikov *et al.* 2001, among others). Unlike inherent anomalies that may occur anywhere in a component at a frequency that is proportional to the volume of material, induced anomalies occur at finite locations such as slots and bolt holes at a frequency that is proportional to the combined area of the machined component features (Enright *et al.* 2004). FAA Advisory Circular 33.14-1 will be updated to include treatment of induced anomalies as the data and methods become available.

A probabilistic methodology has been developed to quantify the risk associated with inherent and induced anomalies (Leverant *et al.* 2000, McClung *et al.* 2004, Enright *et al.* 2004, 2005). The methodology was based on the assumption that engine disk material anomalies occur relatively infrequently (i.e., no more than one significant anomaly may be present in a component), and can be used to assess compliance with the FAA target risk of 1x10^{-9} failures per flight cycle (Federal Aviation Administration 2001). It addresses the influences of primary random variables such as initial anomaly size, applied stresses, and fracture mechanics-related material variables (Wu *et al.* 2002). Additional factors are considered, including the influences of the quantity and quality of nondestructive inspection on overall risk.

The crack formation life (i.e., number of cycles required for an anomaly to form a growing crack) can also be treated as a random variable. Some anomaly types (e.g., titanium hard alpha) have negligible formation lives, and form growing cracks almost immediately (Leverant *et al.* 2000). However, in other materials, the crack formation life may be non-negligible and must be considered in the risk computation (e.g., Grison and Remy 1997, Drar 2001).

Some materials may exhibit relatively higher anomaly occurrence rates compared to those found in premium grade titanium alloys. Components constructed from these materials may have several significant anomalies that form into growing cracks that can ultimately lead to failure. Additional probabilistic considerations are required for the proper treatment of multiple anomaly materials, which is the focus of this paper.

2 RISK ASSESSMENT ASSOCIATED WITH RARE ANOMALIES

2.1 *Inherent material anomalies*

The probability of fracture associated with a rare material anomaly is dependent on both the size and the location of the anomaly within a component. For inherent material anomalies, the uncertainty associated with location can be addressed by subdividing the component into a number of subregions or zones of approximately equal risk. With a zone, the probability of fracture p_i is given by:

$$p_i = P\left(F_{i|A} \bigcap A\right) \tag{1}$$

where event $F_{i|A}$ is fracture failure in zone i given an anomaly in zone i, and event A is the occurrence of an anomaly in zone i.

Since these two events can be considered statistically independent, Eq. (1) can be expressed as:

$$p_i = p_{i|A} \cdot p_A \tag{2}$$

where $p_{i|A}$ and p_A are the probabilities associated with events $F_{i|A}$ and A, respectively. p_A is modeled as a Poisson process with a mean occurrence rate λ_i that is proportional to the volume of material in the zone:

$$\lambda_i = \frac{V_i}{V}\lambda_V \tag{3}$$

where V_i = volume of zone i, V = volume of the component, and λ_V = mean occurrence rate associated with the component volume. The probability of fracture for a component is modeled as a series system of the m zones (Leverant *et al.* 2000, Wu *et al.* 2002):

$$\begin{aligned} p_F &= P[F_1 \cup F_2 \cup \cdots \cup F_m] \\ &= 1 - P\left[\bigcap_{i=1}^{m} \bar{F}_i\right] \end{aligned} \tag{4}$$

If the mean anomaly occurrence rate is relatively small, the probability of more than one significant anomaly in the component is negligible. In this situation, Eqn. (4) reduces to (Freudenthal 1966):

$$p_F \simeq \sum_{i=1}^{m} p_i \tag{5}$$

2.2 *Induced material anomalies*

Induced anomalies are generally located on surface features such as bolt holes and slots. In contrast with inherent anomalies, the mean anomaly occurrence rate associated with induced anomalies λ_i is proportional to the surface area A_i of the feature:

$$\lambda_i = \frac{A_i}{A}\lambda_S \tag{6}$$

where λ_S = anomaly occurrence rate on a per unit surface area basis, and A = reference surface area.

If the component is modeled as a series system consisting of the features that may contain induced anomalies, then Eqns (4) and (5) can be used to estimate the probability of fracture.

2.3 *Life limiting location*

Since crack growth life is dependent on the applied stresses and the geometry of the crack (and associated boundary conditions), it is necessarily dependent on the anomaly location. Cracks usually experience the highest growth rate *da/dN* in regions of relatively high stress and low constraint, leading to relatively lower crack growth life *N*:

$$N = \int_{a_o}^{a_f}\left[\frac{da}{dN}\right]^{-1} da \tag{7}$$

where a_o and a_f are the initial and final crack sizes.

The life limiting location is defined as the position of an anomaly within a zone that results in the lowest deterministic crack growth life value for a specified anomaly size. Within a zone, the conditional probability of fracture is bounded by the probability of fracture at the life limiting location:

$$P\left(F_{i|A}\right) \le P\left(F_{i|A,L}\right) \tag{8}$$

or

$$p_{i|A} \le p_{i|A,L} \tag{9}$$

where event $F_{i|A,L}$ is the fracture of a component given that an anomaly is placed at the life limiting location in the zone, and $p_{i|A,L}$ is the probability associated with this event. Eqn. (9) provides a conservative (upper bound) estimate for the conditional probability of fracture associated with a zone. As the number of zones is increased, the value of $p_{i|A}$ approaches $p_{i|A,L}$.

3 RISK ASSESSMENT ASSOCIATED WITH LARGE NUMBERS OF ANOMALIES

Consider a zone that has multiple anomalies. The probability of fracture within the zone can be expressed as:

$$\begin{aligned} p_i &= P\left[\begin{array}{l}\left(F_{i|A_1}\bigcap A_1\right)\bigcup\left(F_{i|A_2}\bigcap A_2\right)\\ \cdots\bigcup\left(F_{i|A_{n-1}}\bigcap A_{n-1}\right)\bigcup\left(F_{i|A_n}\bigcap A_n\right)\end{array}\right] \\ &= P\left[1-\bigcap_{j=1}^{n}\left(\overline{F}_{i|A_j}\bigcap A_j\right)\right] \end{aligned} \tag{10}$$

or

$$\overline{p}_i = P\left(\overline{F}_i\right) = P\left[\bigcap_{j=1}^{n}\left(\overline{F}_{i|A_j}\bigcap A_j\right)\right] \tag{11}$$

where event $F_{i|A_j}$ is the fracture failure associated with zone *i* given that *j* anomalies are present, and A_j is the occurrence of *j* anomalies in a zone.

Noting that the events A_j are mutually exclusive and collectively exhaustive, if events $F_{i|A_j}$ and A_j are independent, then Eqn. (10) can also be expressed as:

$$p_i = \sum_{j=1}^{n} P\left(F_{i|A_j}\right)\cdot P\left(A_j\right) \tag{12}$$

The probability of fracture of a component is equal to the probability union of the zones. If Eqn. (11) is substituted into Eqn. (4), the following expression is obtained:

$$p_F = 1 - P\left\{\bigcap_{i=1}^{m}\left[\bigcap_{j=1}^{n}\overline{F}_{i|A_j}\bigcap A_j\right]\right\} \tag{13}$$

If zone failures are treated as independent events, Eqn. (13) becomes

$$p_F = 1 - \prod_{i=1}^{m}\left(1-p_i\right) \tag{14}$$

In general, zone failures are at least partially correlated, primarily due to the stress values associated with inertia loading. Eqn. (14) provides a conservative estimate of the component failure probability, provided that correlation among zone failures is nonnegative. Note that if the number of anomalies is relatively small, Eqn. (14) reduces to Eqn. (5).

3.1 *Multiple anomalies at life limiting location*

When multiple anomalies are present, the conditional probability of fracture of a zone can be modeled as a series system consisting of the failure associated with each anomaly:

$$P\left(F_{i|A_j}\right) = 1 - P\left(\bigcap_{j=1}^{n} \overline{F}_{i|A_j}\right) \tag{15}$$

As previously noted, $F_{i|A_j}$ has an upper bound value associated with the life limiting location as specified in Eqn. (9). If this bound is applied (i.e., assume all anomalies are at the life limiting location), Eqn. (14) becomes:

$$P\left(F_{i|A_j}\right) \le 1 - \left[P\left(\overline{F}_{i|A_j,L}\right)\right]^j \tag{16}$$

3.2 *Poisson distributed anomalies*

Suppose that the occurrence of anomalies can be modeled as a Poisson distribution (Haldar and Mahadevan 2000):

$$P\left(A_j\right) = \frac{\left(\lambda_i\right)^j}{j!} \exp\left(-\lambda_i\right) \tag{17}$$

where λ_i is the mean anomaly occurrence rate for the zone.

Substituting the expressions in Eqns. (16) and (17) into Eqn. (12), the probability of fracture for a zone becomes:

$$p_i = \sum_{j=1}^{n}\left[\left\{1 - \left[P\left(\overline{F}_{i|A_j,L}\right)\right]^j\right\} \cdot \frac{\left(\lambda_i\right)^j}{j!}\exp\left(-\lambda_i\right)\right] \tag{18}$$

which can also be expressed as (Roth 1998):

$$p_i = 1 - \exp\left[-\lambda_i \cdot p_{i|A,L}\right] \tag{19}$$

Substituting this expression for p_i in Eqn. (14), the component probability of fracture becomes:

$$p_F = 1 - \prod_{i=1}^{m} \exp\left[-\lambda_i \cdot p_{i|A,L}\right] \tag{20}$$

4 APPLICATION TO COMPRESSOR DISK OF COMMERCIAL AIRCRAFT GAS TURBINE ENGINE

The probabilistic fracture mechanics methodology is illustrated for the aircraft gas turbine engine compressor disk shown in Fig. 1. Internal stresses and temperatures are determined using finite element analysis results, and the disk has a design life of 20,000 flight cycles. Deterministic crack growth life is computed using stress intensity factor solutions for cracks in rectangular plates. For probabilistic computations, the component is subdivided into 31 zones (Fig. 1). Surface and subsurface anomalies are placed at the life limiting locations of the interior and exterior zones, respectively.

4.1 *Limit state*

Failure within a zone occurs when the maximum stress intensity factor K exceeds the fracture toughness K_C:

$$g(\boldsymbol{X},\boldsymbol{Y},N) = K_C - K(\boldsymbol{X},\boldsymbol{Y},N) \ < \ 0 \tag{21}$$

where $g(\boldsymbol{X}, \boldsymbol{Y}, N)$ is dependent on the number of flight cycles N, a vector of input variables unrelated to inspections $\boldsymbol{X}$; and a vector of input variables related to inspections $\boldsymbol{Y}$.

4.2 *Random variables*

Three $\boldsymbol{X}$ random variables are considered for use in crack growth life computations, including initial anomaly size, applied stress, and material properties. The initial anomaly size random variable X_1 is typically a dominant random variable, often characterized as an exceedance curve (e.g., hard alpha anomalies, Fig. 2). It can be expressed as a cumulative distribution function (CDF) (Wu *et al.* 2002) as follows:

$$F_{X_1}(a) = \begin{cases} 0 & a < a_{\min} \\ 1 - \dfrac{D(a) - D(a_{max})}{D(a_{min}) - D(a_{max})} & a_{\min} \le a \le a_{\max} \\ 1 & a > a_{\max} \end{cases} \tag{22}$$

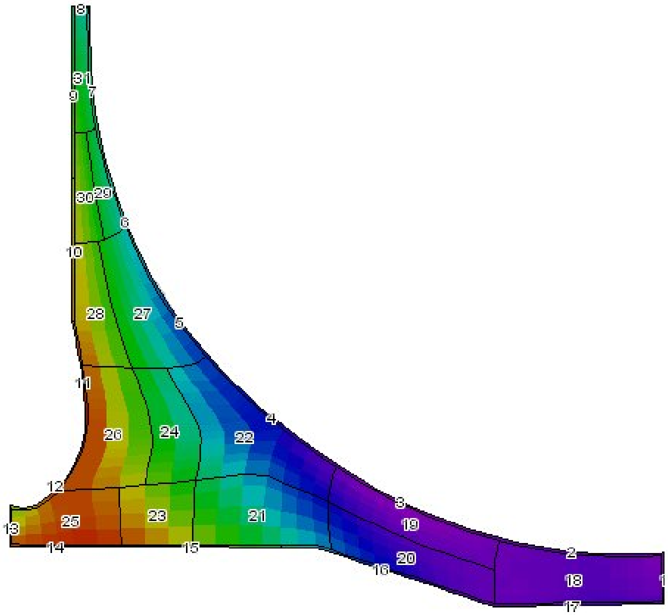


Figure 1. Axisymmetric finite element model of a gas turbine compressor disk subdivided into zones.

Applied stress is modeled as the product of deterministic stress S_{FEM} (often obtained from finite element analysis) and a stress scatter variable X_2:

$$S = X_2 \; S_{FEM} \tag{23}$$

The crack growth life is expressed in terms of the deterministic life N_{NOM} and a life scatter factor X_3 that accounts for material variability:

$$N = X_3 \; N_{NOM} \tag{24}$$

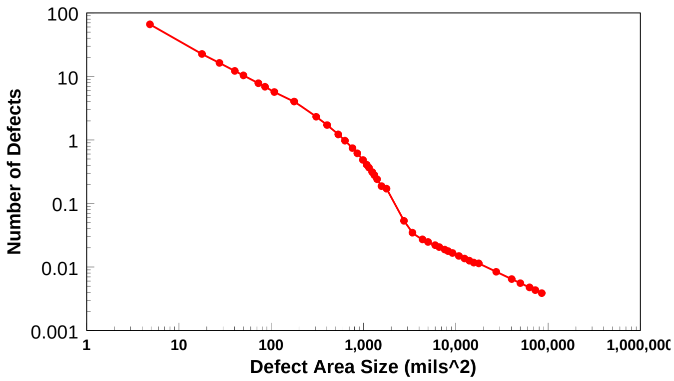


Figure 2. Distribution function of titanium inherent material anomalies expressed in exceedance curve format (Federal Aviation Administration 2001).

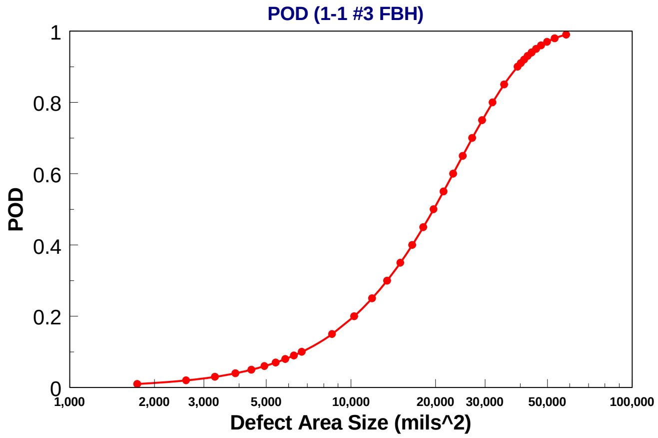


Figure 3. Probability of detection random variable (Federal Aviation Administration 2001).

The Y random variables are the inspection (shop visit) times and the probability of detection (POD). The probability of detecting an anomaly from a population of anomalies (P_{det}) is:

$$P_{det} = \int_0^\infty POD(a) \cdot f(a)\, da \tag{25}$$

where *POD(a)* is the probability of detecting an anomaly with a size (area) greater than *a* (Fig. 3) and *f(a)* is the probability density function associated with an anomaly of size *a*.

4.3 *Component risk results associated with rare anomalies*

Probability of fracture values associated with the gas turbine engine compressor disk (Fig. 1) were computed using a combined technique of numerical integration and importance sampling (Huyse and Enright 2003). Numerical results associated with rare material anomalies (Fig. 2) are indicated in Table 1. For the risk critical zone, the range of values for the mean anomaly occurrence rate is on the order of 10^{-7} to 10^{-5}, and the range of values for the conditional probability of fracture is on the order of 10^{-4} to 10^{-2}. It can be observed that the majority of the risk is concentrated in 3-4 of the 31 zones.

4.4 *Influence of multiple anomalies on probability of fracture*

The results indicated in Table 1 are based on anomaly occurrence rates associated with hard alpha titanium anomalies. For these rare anomalies, component risk can be computed using Eqn. (5), where it is assumed that there is no more than one significant anomaly in the component. However, as anomalies become more plentiful, it becomes increasingly likely that more than one significant anomaly will be present in the disk, and Eqn. (5) is no longer valid.

In this situation, component risk could be computed using Eqns. (14) and (18) for a finite number of anomalies or Eqn. (20) for an infinite number of anomalies. Normalized risk values based on use of these equations are shown in Fig. 4. In Fig. 4(a), it can be observed for this example that the component risk results are nearly identical for all three approaches when the mean zone anomaly occurrence rate is less than about 10^{-3}. Since the zone anomaly occurrence rate for titanium ranges from10^{-7} to 10^{-5}, all of these approaches are valid for this anomaly type. As shown in Figs. 4(a) and (b), the results do

Table 1. Probability of fracture results associated with rare inherent material anomalies for risk critical zones of gas turbine engine compressor disk.

Zone	$p_{i\|A}$	λ_i	p_i	p_i / p_F (%)
6	2.00E-04	4.19E-06	8.39E-10	0.2
8	4.00E-04	9.59E-07	3.84E-10	0.1
9	2.00E-04	8.73E-06	1.75E-09	0.4
10	3.29E-03	5.31E-06	1.75E-08	4.3
11	2.19E-02	3.05E-06	6.67E-08	16.3
12	1.95E-02	1.94E-06	3.79E-08	9.3
13	5.41E-03	5.64E-07	3.05E-09	0.7
14	2.37E-02	1.17E-06	2.77E-08	6.8
15	9.98E-04	2.39E-06	2.39E-09	0.6
21	1.00E-04	5.32E-05	5.32E-09	1.3
23	4.02E-04	2.76E-05	1.11E-08	2.7
25	3.51E-03	3.16E-05	1.11E-07	27.2
26	1.40E-03	6.92E-05	9.69E-08	23.8
28	2.00E-04	5.80E-05	1.16E-08	2.8
30	2.99E-04	3.58E-05	1.07E-08	2.6
31	1.00E-04	3.84E-05	3.84E-09	0.9

not differ substantially unless the anomaly rate is in the range of 10^{-1} to 10^{2}, an increase of roughly six to seven orders of magnitude.

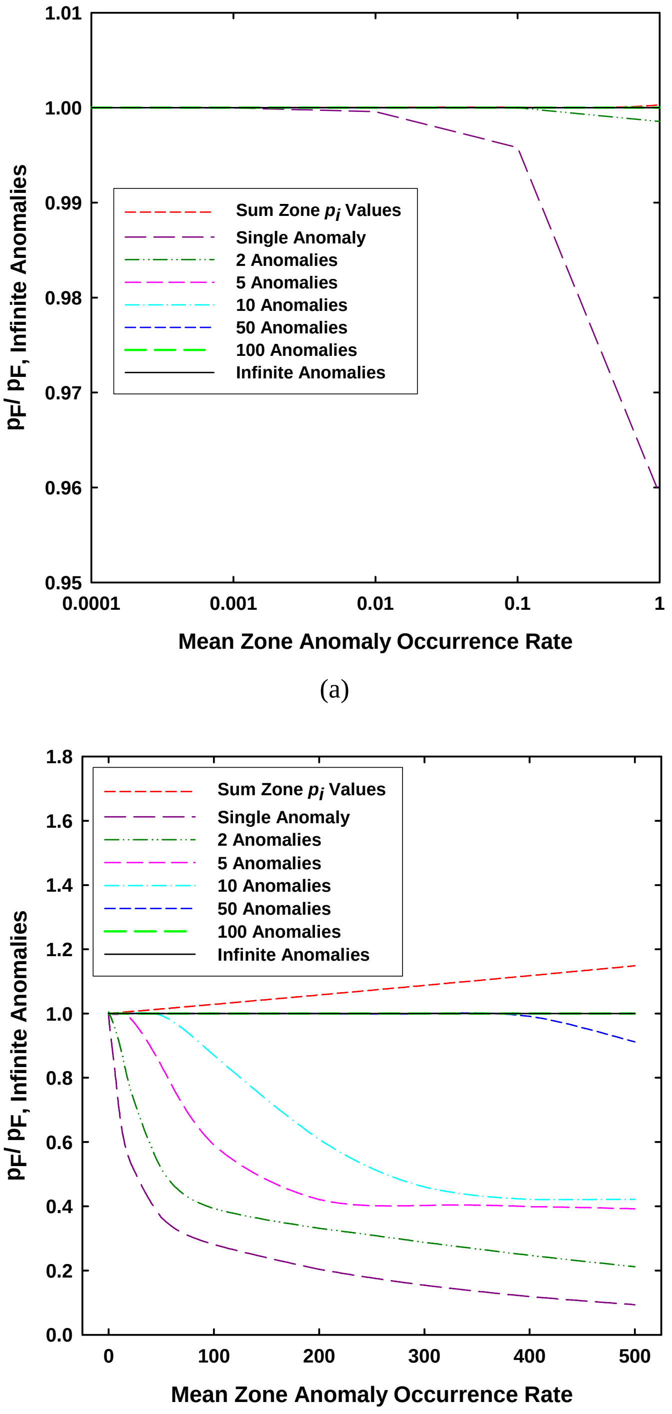
The results shown in Fig. 4 are limited to the compressor disk example for a specific range of values for the conditional probability of fracture. To generalize the results, consider next a fictitious component consisting of 100 zones that have identical values for the anomaly occurrence rate and conditional probability of fracture. For this component, the ratio of the component risk computed using Eqns. (5) and (20) can be expressed as:

$$\frac{\sum_{i=1}^{m} p_{i|A,L}\cdot p_A}{1-\prod_{i=1}^{m}\exp\left[-\lambda_i\cdot p_{i|A,L}\right]}\simeq\frac{m\lambda_i p_{i|A,L}}{1-\exp\left[-\lambda_i\cdot p_{i|A,L}\right]^m} \qquad (26)$$

The influences of anomaly occurrence rate and conditional probability of fracture on component risk evaluated using Eqn. (26) is illustrated in Fig. 5. In Fig. 5(a), it can be observed that when the conditional probability of fracture is relatively large, an anomaly occurrence rate of 10^{-2} or greater will have an influence on the results. As shown in Fig. 5(b), if the conditional probability of fracture is reduced, the anomaly occurrence rate must be on the order of 1 or greater to influence the results.

5 CONCLUSIONS

In this paper, a methodology was presented for risk assessment of components with large numbers of material anomalies. The methodology was illustrated for an aircraft gas turbine engine component, including the relative influences of the zone anomaly occurrence rate and conditional probability of fracture on overall component risk. The results can be used to extend the existing zone-based probabilistic framework for titanium materials (with hard alpha anomalies) for application to other alloys with a potentially large number of material anomalies.



(a)

(b)

Figure 4. Influence of number of anomalies on normalized component risk values: (a) relatively rare anomalies, and (b) relatively frequent anomalies.

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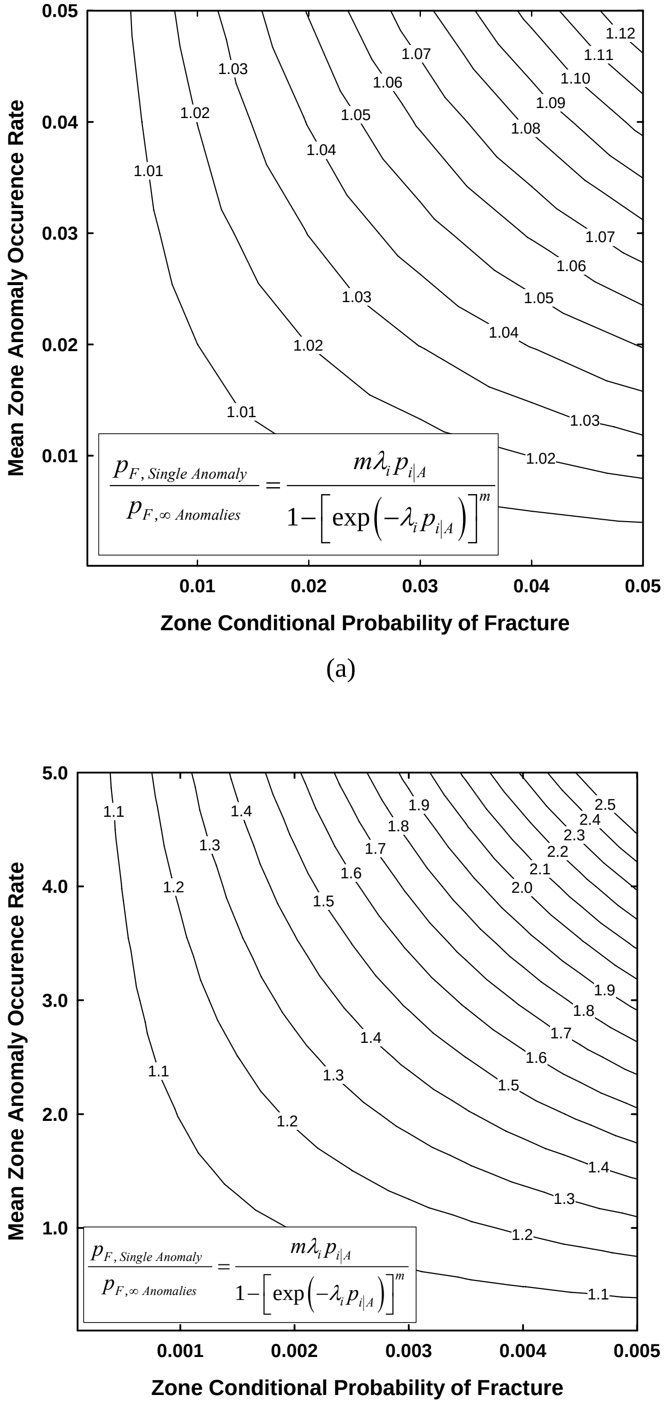

(a)

(b)

Figure 5. Influence of zone anomaly occurrence rate λ_i and conditional probability of fracture $p_{i|A}$ on normalized component risk associated for $p_{i|A}$ values with relative magnitudes: (a) 10^{-2}, and (b) 10^{-3}.

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